III. "On the Application of Parabolic Trigonometry to the Investigation of the Properties of the Common Catenary." By the Rev. JAMES BOOTH, LL.D., F.R.S. Received March 19, 1857.

Some time ago, on the publication of a paper read by me last summer at Cheltenham before the Mathematical Section of the British Association on Parabolic Trigonometry and the Geometrical origin of Logarithms, Sir John Herschel called my attention to the analogy which exists between the equation of the common catenary referred to rectangular coordinates, and one of the principal formulæ of parabolic trigonometry. Since that time I have partially investigated the subject, and find, on a very cursory examination, that the most curious analogies exist between the properties of the parabola and those of the catenary,—that in general for every property of the former a corresponding one may be discovered for the latter. In this paper I cannot do more than give a mere outline of these investigations, but I hope at some future time, when less occupied with other avocations than at present, I may be permitted to resume the subject. I will only add, that the properties of this curve appear to be as inexhaustible as those of the circle or any other conic section.

II. The equations of the common catenary referred to rectangular coordinates are

$$y=\frac{m}{2}\left(e^{\frac{x}{m}}+e^{-\frac{x}{m}}\right), \qquad s=\frac{m}{2}\left(e^{\frac{x}{m}}-e^{-\frac{x}{m}}\right).$$

The point O may be called the focus, whose distance to the vertex A of the curve is $=m$.

It will simplify the investigations, without lessening their generality, if we assume the modulus or focal distance $m=1$.

Assume $2\sec\theta=e^{x}+e^{-x}, \quad 2\tan\theta=e^{x}-e^{-x}$ . . . . . . (1.)

Then $y=\sec\theta, \quad s=\tan\theta$ . . . . (2.)

Now if we make x', x'', x''', &c. successively equal to $2x$, $3x$, $4x$, &c., we shall have (see 'Parabolic Trigonometry,' No. XXVI.)

$$\begin{array}{ll}
y\ \ =\sec\ \theta & s\ \ =\tan\ \theta \\
y'\ =\sec(\theta\perp\theta) & s'\ =\tan(\theta\perp\theta) \\
y''\,=\sec(\theta\perp\theta\perp\theta) & s''\,=\tan(\theta\perp\theta\perp\theta) \\
y'''=\sec(\theta\perp\theta\perp\theta\perp\theta) & s'''=\tan(\theta\perp\theta\perp\theta\perp\theta)
\end{array}$$

Now ('Parabolic Trigonometry,' No. III.) it has been shown that

$$(\sec\theta+\tan\theta)^n=\sec(\theta \perp \theta \perp \text{to } n \text{ terms})+\tan(\theta \perp \theta \perp \text{to } n \text{ terms});$$

hence in the catenary we shall have

$$(y+s)^n=(y_{,,,}\ldots n+s_{,,,}\ldots n),$$

or *if two points on the catenary be assumed, the abscissa of one being* n *times that of the other, the* nth *power of the sum of the ordinate and arc of the latter will be equal to the ordinate and arc of the former.*

We may graphically exhibit with great simplicity the sum of a series of angles added together by the parabolic or logarithmic plus $\perp$.

Let a set of equidistant ordinates—for simplicity let the common interval be unity—meet the catenary in the points b, c, d, k, l, and then let the catenary be supposed to be stretched along the horizontal tangent passing through the vertex A. Let the points b, c, d, k, l, on the catenary in its free position, coincide with the points $\beta, \gamma, \delta, \kappa, \lambda$, on the horizontal line when strained in that position, and as x or Aβ, Aγ, Aδ, Aκ, Aλ is successively equal to

$$m,\ 2m,\ 3m,\ 4m, \text{ \&c. or to } 1,\ 2,\ 3,\ 4, \text{ \&c. if } m=1,$$

we shall have

$$\begin{array}{ll} 2y\ \ =e^1+e^{-1} & s\ \ =e^1-e^{-1} \\ 2y_{,}\ =e^2+e^{-2} & s_{,}\ =e^2-e^{-2} \\ 2y_{,,}=e^3+e^{-3} & s_{,,}=e^3-e^{-3} \\ 2y_{,,,}=e^4+e^{-4} & s_{,,,}=e^4-e^{-4} \end{array}$$

Hence the angle AOβ or ε is such that

$$\sec\varepsilon+\tan\varepsilon=e,$$

AOγ such that, $\sec \text{AO}\gamma+\tan \text{AO}\gamma=e^2$, or $\text{AO}\gamma=\varepsilon\perp\varepsilon$

AOδ such that, $\sec \text{AO}\delta+\tan \text{AO}\delta=e^3$, or $\text{AO}\delta=\varepsilon\perp\varepsilon\perp\varepsilon$;

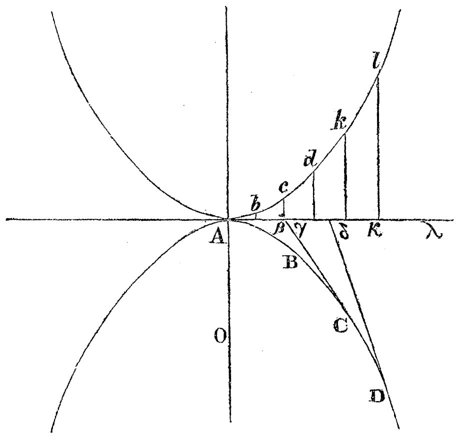
consequently if we draw lines from the focus O successively to the points $\beta, \gamma, \delta, \kappa, \lambda$, the angles AO$\beta$, AO$\gamma$, AO$\delta$, AO$\kappa$, AO$\lambda$ will be the angles ε, $\varepsilon\perp\varepsilon$, $\varepsilon\perp\varepsilon\perp\varepsilon$, $\varepsilon\perp\varepsilon\perp\varepsilon\perp\varepsilon$.

This is one of the simplest graphical representations we can have of angles added together by the parabolic or logarithmic plus $\perp$.

Hence as successive multiples of an arc of a circle give successive arithmetical multiples of the corresponding angle at the centre, so successive multiples of a given abscissa give successive arcs of the

catenary which, extended along the vertical tangent, subtend at O successive parabolic multiples of the original angle.

Since the original interval was assumed equal to m, and as the arc



of the catenary is always longer than the abscissa which subtends it, or $A\beta > Ab$, it follows, as has been shown in 'Parabolic Trigonometry,' No. XXII., that $\varepsilon > 45^\circ$.

Since $\frac{dy}{dx}=\frac{1}{2}(e^x-e^{-x})$ we shall have $\frac{dy}{dx}=\tan\theta$, but as $\frac{dy}{dx}$ is the trigonometrical tangent of the angle which the linear tangent at the point (xy) makes with the axis of the abscissa, hence this other theorem :

Let a set of equidistant ordinates meet the catenary in the points b, c, d, k, l, *and at these points let tangents to the curve be drawn, they will be inclined to the axis of the abscissa by the angles* θ, $\theta\perp\theta$, $\theta\perp\theta\perp\theta$, $\theta\perp\theta\perp\theta\perp\theta$, &c., which is even a yet simpler geometrical representation than the preceding.

Hence also it evidently follows that as the limit of the angle which a tangent to the catenary makes with the axis of the abscissa is a right angle, the limit of the angle $\theta\perp\theta\perp\theta\perp\theta\perp\theta$, *ad infinitum,* is a right angle.

We have also this other theorem :

If with the point O *as focus, and* A *as vertex, we describe a parabola, and from the points* β, γ, δ, κ, λ *we draw tangents* βB, γC,

δD, κK, λL *to the parabola, the differences between these tangents and the corresponding parabolic arcs, namely,* AD, AC, AD, AK, AL, *will be* m, $2m$, $3m$, $4m$, *or*

$$AB-\beta B=m,\quad AC-\gamma C=2m,\quad AD-\delta D=3m,\quad AK-\kappa K=4m \text{ \&c.}$$

This is evident (see 'Parabolic Trigonometry,' No. XXVI.) for the angles $AO\beta=\varepsilon$, $AO\gamma=\varepsilon\perp\varepsilon$, $AO\delta=\varepsilon\perp\varepsilon\perp\varepsilon$, $AO\kappa=\varepsilon\perp\varepsilon\perp\varepsilon\perp\varepsilon$.

We may further extend these properties of the catenary. To simplify the expressions, let $Y\phi$ denote the ordinate of a point on the catenary at which the tangent makes the angle ϕ with the axis of X. Let $S\phi$ denote the arc measured from the lowest point, and let $X\phi$ signify the ordinate.

Then $$S\phi=\tan\phi,\qquad Y\phi=\sec\phi.$$

Now let x, $x_{\prime}$, $x_{\prime\prime}$ be the abscissæ of the three arcs whose tangents make the angles ϕ, χ, ω with the axis of x, and let the equation of condition be simply

$$x_{\prime\prime}=x+x_{\prime}.$$

Then we shall have the following relations between the corresponding arcs and ordinates of the catenary—

$$S\omega=S\phi\, Y\chi+S\chi Y\phi$$
$$Y\omega=Y\phi Y\chi+S\phi S\chi$$

when
$$x_{\prime}=x$$
$$2S^2\phi=Y\omega-1$$
$$2Y^2\phi=Y\omega+1$$
$$Y\omega=Y^4\phi-S^4\phi.$$

Let there be four arcs of the catenary whose abscissæ x, $x_{\prime}$, $x_{\prime\prime}$, $x_{\prime\prime\prime}$ shall be connected by the following relation

$$x_{\prime\prime}=x+x_{\prime}$$
$$x_{\prime\prime\prime}=x_{\prime\prime}+x_{\prime\prime}\ \text{ or }\ x_{\prime\prime\prime}=x+x_{\prime}+x_{\prime\prime}.$$

Let ϖ, ϕ, χ, ψ be the corresponding angles made by the tangents to the extremities of the arcs $S\omega$, $S\phi$, $S\chi$, $S\psi$.

Then we shall have the following relations between the arcs and the ordinates—

$$S\varpi=S\phi Y\chi Y\psi+S\chi Y\phi Y\psi+S\psi Y\phi Y\chi+S\phi S\chi S\psi$$
$$Y\varpi=Y\phi Y\chi Y\psi+Y\phi S\chi S\psi+Y\chi S\phi S\psi+Y\psi S\phi S\chi.$$

Hence also
$$\frac{S\omega}{Y\phi Y\chi Y\psi}=\left(\frac{S\phi}{Y\phi}\right)+\left(\frac{S\chi}{Y\chi}\right)+\left(\frac{S\psi}{Y\psi}\right),$$

or the ratio of the fourth arc to the product of the ordinates of the three preceding arcs is equal to the sum of the ratios of each preceding arc to its ordinate.

We have also

$$\frac{Y\bar{\omega}}{Y\phi Y\chi Y\psi}=1+\left(\frac{S\phi S\chi}{Y\phi Y\chi}\right)+\left(\frac{S\chi}{Y\chi}\frac{S\psi}{Y\psi}\right)+\left(\frac{S\psi}{Y\psi}\frac{S\phi}{Y\phi}\right).$$

Let $\qquad x=x_{,}=x_{,,}$, and $x_{,,,}=3x$,

then we shall have $\qquad S\omega=4S^3\phi+3S\phi$,

an equation which gives the relation between two arcs of the catenary, the abscissa of the one being equal to three times that of the other.

When one abscissa is double of the other, the arcs are related by the equation $2YS=S_{,}$.

Since $\qquad \sin^2\phi=\frac{\sec(\phi\perp\phi)-1}{\sec(\phi\perp\phi)+1}$,

and $\sin\phi=\frac{S}{Y}$, we shall have

$$\frac{S^2}{Y^2}=\frac{Y_{,}-1}{Y_{,}+1},$$

an equation which enables us to calculate $Y_{,}$ when we know Y, since $S^2=Y^2-1$. Thus the catenary may be constructed by points.

Let $s, y, s_{,}y_{,}, s_{,,}y_{,,}, s_{,,,}y_{,,,}$ be four arcs and corresponding ordinates of a catenary, whose abscissæ are connected by the equation

$$x_{,,,}=x_{,,}+x_{,}+x,$$

then we shall have

$$\frac{s_{,,,}}{y_{,,,}}=\frac{\frac{s}{y}+\frac{s_{,}}{y_{,}}+\frac{s_{,,}}{y_{,,}}+\frac{ss_{,}s_{,,}}{yy_{,}y_{,,}},}{1+\frac{ss_{,}}{yy_{,}}+\frac{s_{,}s_{,,}}{y_{,}y_{,,}}+\frac{s.s_{,,}}{yy_{,,}}.}$$

The Society then adjourned over the Easter holidays to Thursday, April 23.